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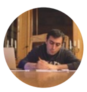

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BOUNDARY BEHAVIOR FOR A SINGULAR PERTURBATION PROBLEM

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Abstract. In this paper we study the boundary behaviour of the family of solutions $\{u^\varepsilon\}$ to singular perturbation problem $\Delta u^\varepsilon = \beta_\varepsilon(u^\varepsilon), |u^\varepsilon| \leq 1$ in $B_1^+ = \{x_n > 0\} \cap \{|x| < 1\}$, where a smooth boundary data f is prescribed on the flat portion of ∂B_1^+. Here $\beta_\varepsilon(\cdot) = \frac{1}{\varepsilon}\beta\left(\frac{\cdot}{\varepsilon}\right), \beta \in C_0^\infty(0,1), \beta \geq 0, \int_0^1 \beta(t) = M > 0$ is an approximation of identity. If $\nabla f(z) = 0$ whenever $f(z) = 0$ then the level sets $\partial\{u^\varepsilon > 0\}$ approach the fixed boundary in tangential fashion with uniform speed. The methods we employ here uses delicate analysis of local solutions, along with elaborated version of the so-called monotonicity formulas and classification of global profiles.

To Juan-Luis Vazquez on the occasion of his 70th Birthdate.

1. Introduction

In this paper we study the boundary behaviour of the family of solutions $\{u^\varepsilon\}$ to singular perturbation problem

$$\begin{cases} \Delta u^\varepsilon = \beta_\varepsilon(u^\varepsilon), & \text{in } B_1^+, \\ u^\varepsilon = f, & \text{on } B_1', \\ |u^\varepsilon| \leq 1, & \text{in } B_1^+, \end{cases} \tag{1.1}$$

in the half unit ball $B_1^+ = \{x_n > 0\} \cap \{|x| < 1\}$, where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and $B_1' = B_1 \cap \{x_n = 0\}$. The perturbed right hand side β_ε, satisfies certain conditions that are specified below. Also, the boundary data f is a smooth function satisfying the following condition (specially on the flat portion of

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the boundary)

$$\nabla f(z) = 0 \text{ whenever } f(z) = 0. \tag{1.2}$$

Under these conditions we show that close to a "touching" point between the free and the fixed boundary, the free boundary touches the fixed one in a uniformly tangential fashion. Here free boundary refers to the zero level surface of our solution, $\partial\{u^\varepsilon > 0\}$.

Our analysis is based on utilization of the monotonicity formula and classification of global/blow-up solutions. The analogous problem for minimisers of the functional

$$J(u) = \int_{B_1^+} |\nabla u|^2 + \lambda_+ \chi_{\{u>0\}} + \lambda_- \chi_{\{u\leq 0\}}$$

is studied in [7], where $\lambda_+^2 - \lambda_-^2 > 0$.

Problem (1.1) appears in the mathematical theory of combustion as a model with high activation energy, which is of order $\frac{1}{\varepsilon}$, in an ε-strip approximation of the flame, see [9] Chapter 4.3. The family $\{\beta_\varepsilon(\cdot)\}$ renders such approximation (see (1.3) below). Also, for more recent mathematical treatment see [2, 3, 4] and references therein.

Problem set-up and Standing Assumption:

To fix the ideas we suppose that

$$\beta_\varepsilon(\cdot) = \frac{1}{\varepsilon}\beta\left(\frac{\cdot}{\varepsilon}\right), \quad \beta \in C_0^\infty(0,1), \quad \beta \geq 0, \quad \int_0^1 \beta(t)dt = M > 0. \tag{1.3}$$

Observe that by definition of $\beta_\varepsilon(t)$ we have

$$\int_0^\varepsilon \beta_\varepsilon(t)dt = \int_0^1 \beta(t)dt = M > 0.$$

The limit function, obtained as $\varepsilon \to 0$ solves locally the following free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \ \{u > 0\} \cup \{u < 0\}, \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 2M & \text{on } \ \partial\{u > 0\}. \end{cases} \tag{1.4}$$

in a very weak sense, see [2], [3, 4].

Let f be a smooth function on $\{x_n = 0\} \cap B_1$ such that (1.2) is satisfied. It is known that under (1.2) the family $\{u^\varepsilon\}$ is uniformly bounded in Lipschitz norm [6]

$$\sup_{x\in B^+_{1/2}} |\nabla u^\varepsilon(x)| \le L, \tag{1.5}$$

with a positive constant $L > 0$, which is independent of ε for any solution of (1.1).

Assumptions (1.3) are standard (see [2], [6]), however one can relax the assumption $\beta \in C^\infty_0(0,1)$ to $\beta \in C^{0,1}_0(0,1)$ in the proof of the Lipschitz norm estimate (1.5).

Non-degeneracy: Throughout the paper we shall assume a linear non-degeneracy at the origin, standard for such problems, which is

$$\int_{B^+_r} u \ge C_0 r^{n+1}, \tag{1.6}$$

for a universal C_0.

Remark 1.1. *If large enough negative and positive phases are present then one can prove that u^+ is non-degenerate. Namely, let $x_0 \in \partial\{u > 0\}$, if there is a unit vector e, such that*

$$\begin{aligned}\liminf_{r\to 0} \frac{|\{u>0\}\cap\{(x-x_0)\cdot e > 0\}\cap B_r(x_0)|}{|B_r|} &= \alpha_1 \\ \liminf_{r\to 0} \frac{|\{u<0\}\cap\{(x-x_0)\cdot e < 0\}\cap B_r(x_0)|}{|B_r|} &= \alpha_2\end{aligned} \tag{1.7}$$

with $\alpha_1 + \alpha_2 > \frac{1}{2}$ then there exists a tame constant $C > 0$ such that $\sup_{B_r(x_0)} u^\varepsilon \ge Cr$ [4] *Theorem 6.3.*

Our main result is the following theorem, stating tangential touch between the free and fixed boundary.

Theorem 1. *Let u^ε be a solution to our problem* (1.1)*, satisfying non-degeneracy* (1.6)*, and suppose f, β, satisfy the assumptions above. Then, there are $\varepsilon_0 > 0$, a radius $r_0 > 0$, and a modulus of continuity $\sigma(x)$ depending on f, n, M and L such that*

$$\partial\{u^\varepsilon > 0\} \cap B^+_{r_0} \subset \{x_n < \sigma(|x|)|x|\} \qquad \forall \varepsilon \in (0, \varepsilon_0).$$

It seems plausible that when $f \equiv 0$, one should obtain stronger result, such as the free boundary is locally a $C^{1,\alpha}$-graph, close to touching points. Such a result needs more careful analysis of the local problem, and techniques will depend strongly on the choice of β.

2. Technicalities

In this section we gather a few standard results that are needed for our analysis of problem (1.1).

Proposition 2.1. *Let* $v_j = \frac{u^{\varepsilon_j}(r_j x)}{r_j}$, *with* u^{ε_j} *being a solution of* (1.1). *Then, after passing to a subsequence, there exists* v *so that*

(i) $v_j \to v$ *uniformly on compact subsets of* $\mathbb{R}^n_+$ *and in* $C^{0,\alpha}(\overline{B^+_R}), 0 < \alpha < 1$, *for each* $R > 0$,

(ii) *for each* R, $v_j \rightharpoonup v$ *weakly in* $W^{1,2}(B^+_R)$,

(iii) $\nabla v_j(x) \to \nabla v(x)$ *for a.e.* x.

Next we introduce the Alt-Caffarelli-Friedman monotonicity formulae and state its properties [1].

Lemma 2.2. *Let* h_1, h_2 *be two non-negative continuous sub-solutions of* $\Delta u = 0$ *in* $B(x^0, R)$ $(R > 0)$. *Assume further that* $h_1 h_2 = 0$ *and that* $h_1(x^0) = h_2(x^0) = 0$, *and set (for* $0 < r < R$*)*

$$\varphi(r) = \varphi(r, h_1, h_2, x^0) = \frac{1}{r^4}\left(\int_{B(x^0,r)} \frac{|\nabla h_1|^2\ dx}{|x - x^0|^{n-2}}\right)\left(\int_{B(x^0,r)} \frac{|\nabla h_2|^2\ dx}{|x - x^0|^{n-2}}\right).$$

Then

$$\frac{d}{dr}\varphi(r) \geq \frac{2\varphi(r)}{r} A_r, \tag{2.1}$$

where $A_r > 0$ *is given by (see [1] Lemmas 2.2-2.3)*

$$\sqrt{A_r} = \frac{C_n}{r^{n-1}} Area\left(\partial B_r \setminus (supp\ h_1 \cup supp\ h_2)\right). \tag{2.2}$$

Lemma 2.3. *Let* w *be a solution of* $\Delta v = \beta(v)$ *in* $\mathbb{R}^n_+$ *such that* $|\nabla v(x)| \leq L$, *in* $\mathbb{R}^n_+$, *and* $v = 0$ *on* $\{x_n = 0\}$. *Then either* $v \equiv 0$ *or* $v > 0$.

Proof. Let $h_1 = \max(v, 0)$ and $h_2 = -\min(v, 0)$, then h_1, h_2 are nonnegative subharmonic functions. Then applying Lemma 2.2 we see that for $\sqrt{A_r}$ (2.2) we have the lower bound

$$\sqrt{A_r} \geq \frac{C_n}{r^{n-1}} \frac{n}{2} |B_1| r^{n-1} \geq C_n > 0.$$

Thus $\varphi(r) \geq \varphi(r_0)\left(\frac{r}{r_0}\right)^{C_n} \to \infty$ if $r \to \infty$ for fixed $r_0 > 0$. On the other hand from $|\nabla v(x)| \leq L$, in $\mathbb{R}^n_+$, it follows that $\varphi(r) \leq (n|B_1|)^2 L^4$. Thus either $h_2 \equiv 0$ or $h_1 \equiv 0$. The latter cannot be true unless $v \equiv 0$ since v is subharmonic in $\mathbb{R}^n_+$. Thus $h_2 \equiv 0$ yielding $v \geq 0$. To finish the proof we have to show that if v is not a trivial solution then $v > 0$. Because $\beta \geq 0$ and $\beta \in C^\infty_0(0,1)$ it follows that there is $K > 0$ large such that $Ks - \beta(s) \geq 0$ for any $s \geq 0$. Indeed, one can take the concave envelope of the graph of β and choose $K > 0$ to be the largest slope of the supporting lines. Thus we have

$$\Delta v - Kv = \beta(v) - Kv \leq 0$$

and by applying the strong maximum principle ([5] Theorem 3.5) it follows that $v > 0$ in $\mathbb{R}^n_+$. $\square$

An important tool to be used in the classification of global solutions is Weiss' monotonicity formula, which is based on a Pohozhaev type identity.

Lemma 2.4. *Let v be a solution of $\Delta v = \beta_\varepsilon(v)$. Then for any $\psi \in C^{0,1}_0(\mathbb{R}^n, \mathbb{R}^n)$, $\psi(x) = (\psi^1(x), \dots, \psi^n(x))$ there holds*

$$\begin{aligned}\int\limits_{\mathbb{R}^n_+} v_i v_j \psi^i_j &= \int\limits_{\mathbb{R}^n_+} \operatorname{div}\psi\left[\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v)\right] \\ &\quad + \int\limits_{\{x_n=0\}} (\nabla v \cdot \nu)\nabla v \cdot \psi - \int\limits_{\{x_n=0\}} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v)\right)\psi \cdot \nu\end{aligned}$$

where $\mathcal{B}_\varepsilon(t) = \int_0^t \beta_\varepsilon(s)ds$ and $v_i(x) = \partial_{x_j} v(x), \psi^j_i(x) = \partial_{x_i}\psi^j_i$.

Proof. Let us fix a Lipschitz continuous $\psi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\operatorname{supp}\psi \subset B_r$ for some $r > 0$, then we have from the divergence theorem

$$\begin{aligned}2\int\limits_{B^+_r} v_i v_j \psi^i_j &= 2\int\limits_{\partial B^+_r} (\nabla v \cdot \nu)\nabla v \cdot \psi - \int\limits_{B^+_r} (\nabla|\nabla v|^2 + 2\nabla v \Delta v) \cdot \psi \\ &= 2\int\limits_{\partial B^+_r} (\nabla v \cdot \nu)\nabla u \cdot \psi - 2\int\limits_{B^+_r} \left(\nabla\frac{|\nabla v|^2}{2} + \nabla v \beta_\varepsilon(v)\right)\cdot \psi \\ &= 2\int\limits_{\partial B^+_r} (\nabla v \cdot \nu)\nabla v \cdot \psi - 2\int\limits_{B^+_r} \nabla\left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v)\right)\cdot \psi.\end{aligned}$$

Now applying the divergence theorem again we obtain

$$
\begin{aligned}
2\int_{B_r^+} v_i v_j \psi^i_j &= 2\int_{\partial B_r^+\cap\{x_n=0\}} (\nabla v\cdot\nu)\nabla u\cdot\psi - 2\int_{\partial B_r^+\cap\{x_n=0\}}\left(\frac{|\nabla v|^2}{2}+\mathcal{B}_\varepsilon(v)\right)\psi\cdot\nu \\
&\quad +2\int_{B_r^+}\left(\frac{|\nabla v|^2}{2}+\mathcal{B}_\varepsilon(v)\right)\operatorname{div}\psi.
\end{aligned}
$$

Hence the proof is completed. □

The integral identity in Lemma 2.4 allows to construct a functional defined on the upper half balls $B_r^+(x_0)=B_r(x_0)\cap\{x_n>0\}, x_0\in\{x_n=0\}$ which is monotone function of r for any fixed $\varepsilon>0$. This is done by choosing ψ appropriately.

Lemma 2.5. *Let v be a solution $\Delta v=\beta_\varepsilon(v)$ in $B_{r_0}(x_0)$ and $v=g_\varepsilon$ on $\{x_n=0\}\cap B_{r_0}(x_0), x_0\in\{x_0=0\}$ such that $g_\varepsilon\in C^{1,\alpha}$, and $g_\varepsilon(x)=|x|\ell(x)$ with $\ell(x)\le C|x|^\alpha,\alpha>0$. Introduce the functional $W_\varepsilon(r), r<r_0$*

(2.3)

$$
W_\varepsilon(r):=\frac{1}{r^n}\int_{B_r^+}\left(\frac{|\nabla v|^2}{2}+\mathcal{B}_\varepsilon(v)\right)-\frac{1}{2r^{n+1}}\int_{\partial B_r^+\cap\{x_n>0\}}v^2+C\|g_\varepsilon\|_{C^{1,\alpha}(B_{r_0}\cap\{x_n=0\})}r^\alpha,
$$

for some positive large constant $C>0$. Then W_ε is monotone non-decreasing function of r, for any fixed ε. Moreover, for any $0<S<R<r_0$ the following formula holds

$$
W_\varepsilon(R)-W_\varepsilon(S)=\int_S^R\frac{d\rho}{\rho^n}\int_{\partial B_\rho\cap\{x_n>0\}}\left(\nabla v\cdot\nu-\frac{v}{\rho}\right)^2. \tag{2.3}
$$

Proof. Let us take $\psi(x)=x\eta_\delta(x)$ where

$$
\eta_\delta(x):=\begin{cases}1 & \text{if } x\in B_\rho,\\ \frac{\rho+\delta-|x|}{\delta} & \text{if } x\in B_{\rho+\delta}\setminus B_\rho,\\ 0 & \text{if } x\in\mathbb{R}^n\setminus B_{\rho+\delta},\end{cases}
$$

and $\delta>0$ is a small parameter. Then, by direct computation

$$
\partial_{x_j}\psi^i(x):=\begin{cases}\delta_{ij} & \text{if } x\in B_\rho,\\ \delta_{ij}\eta-\frac{1}{\delta}\frac{x_1x_j}{|x|} & \text{if } x\in B_{\rho+\delta}\setminus B_\rho,\\ 0 & \text{if } x\in\mathbb{R}^n\setminus B_{\rho+\delta},\end{cases}
$$

and

$$\operatorname{div}\psi(x) := \begin{cases} N & \text{if } x \in B_\rho, \\ N\eta - \frac{|x|}{\delta} & \text{if } x \in B_{\rho+\delta} \setminus B_\rho, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_{\rho+\delta}. \end{cases}$$

Hence, plugging the last two formulas into the identity of Lemma 2.4 and noting that

$$\int_{\partial B_\rho^+ \cap \{x_n=0\}} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) \eta_\delta (x \cdot \nu) = 0,$$

because $x \cdot \nu = 0$ if $x \in \{x_n = 0\}$, and

$$\int_{\partial B_r^+ \cap \{x_n=0\}} (\nabla v \cdot \nu)(\nabla v \cdot x)\eta_\delta = \int_{\partial B_\rho^+ \cap \{x_n=0\}} (\nabla v \cdot \nu) \left(\sum_{i=1}^{n-1} \partial_{x_i} f_j(x) x_i \right) \eta_\delta \sim \rho^{n+\alpha}$$

because $\nabla v \cdot x$ is the tangential derivative on $\{x_n = 0\}$. We finally obtain

$$\begin{aligned} &\int_{B_\rho^+} |\nabla v|^2 + \int_{B_{\rho+\delta}^+ \setminus B_\rho^+} \nabla u \cdot \left(\nabla v \eta_\delta - \frac{1}{\delta} \nabla v \frac{x \otimes x}{|x|} \right) \\ &\qquad = n \int_{B_\rho^+} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) + \int_{B_{\rho+\delta}^+ \setminus B_\rho^+} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) \left(n\eta_\delta - \frac{|x|}{\delta} \right). \end{aligned}$$

Notice that from Lebesgue's theorem on the absolute continuity of integrals we have that

$$\int_{B_{\rho+\delta}^+ \setminus B_\rho^+} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) n\eta_\delta \to 0, \quad \int_{B_{\rho+\delta}^+ \setminus B_\rho^+} |\nabla v|^2 \eta_\delta \to 0 \quad \text{as } \delta \to 0.$$

Therefore, sending $\delta \to 0$, we end up with the identity

$$\begin{aligned} &\int_{B_\rho^+} |\nabla v|^2 - \rho \int_{\partial B_\rho \cap \{x_n > 0\}} (\partial_\nu v)^2 \\ &\qquad = n \int_{B_\rho^+} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) - \rho \int_{\partial B_\rho \cap \{x_n > 0\}} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) + O(\rho^{n+\alpha}). \end{aligned}$$

One the other hand we have that

$$\int_{B_\rho} |\nabla v|^2 = \int_{\partial B_\rho^+} v \partial_\nu v - \int_{B_\rho^+} v \beta_\varepsilon(v) = \int_{\partial B_\rho^+ \cap \{x_n > 0\}} v \partial_\nu v - O(\rho^{n+\alpha}).$$

Thus multiplying both sides of the last identity by ρ^{-n-1} the proof follows.

□

Next lemma takes care of the limit of W_ε when $\varepsilon \to 0$.

Lemma 2.6. *Let v_ε be as in Lemma 2.5 and $v_\varepsilon \to v$ for a suitable subsequence $\varepsilon = \varepsilon_j$, see Proposition 2.1, then $W_\varepsilon(r) \to W(r)$ where*

$$W(r) = \frac{1}{r^n}\int_{B_r^+}\left(\frac{|\nabla v|^2}{2} + \gamma(x)\right) - \frac{1}{2r^{n+1}}\int_{\partial B_r^+\cap\{x_n>0\}} v^2 + C\|g_0\|_{C^{1,\alpha}(B_{r_0}\cap\{x_n=0\})} r^\alpha$$

$g_0 = \lim_{\varepsilon\to 0} g_\varepsilon$, W_ε is given by (2.3) and $\gamma(x) \in L^\infty$ is the weak-star limit of $\mathcal{B}_\varepsilon(v_\varepsilon)$. Furthermore, for any $0 < S < R < \frac{1}{r_j}$ the following holds

$$W(R) - W(S) = \int_S^R \frac{d\rho}{\rho^n}\int_{\partial B_\rho\cap\{x_n>0\}}\left(\nabla v\cdot\nu - \frac{v}{\rho}\right)^2. \tag{2.4}$$

Proof. Since $\mathcal{B}_\varepsilon \in L^\infty$ then it follows that there is $\gamma(x) \in L^\infty$ such that for any $R > 0$ and $\eta \in L^1(B_R^+)$ there holds

$$\int_{B_R^+}\mathcal{B}_\varepsilon(v_\varepsilon(x))\eta(x)dx \to \int_{B_R^+}\gamma(x)\eta(x)dx,$$

in other words γ is the weak-star limit. Note that $\gamma(x) \geq M\chi_{\{v>0\}}$. Indeed, if $v(x) > 0$ then by uniform convergence there is $\delta > 0$ such that $v_\varepsilon(y) \geq \frac{v(x)}{2}$ if $y \in B_\delta(x)$. Thus $\mathcal{B}_\varepsilon(x) = M$ for sufficiently small ε. To fix the ideas we set $v_j := v_{\varepsilon_j}$ for some $\varepsilon_j \to 0$ such that $v_j \to v$. Thus by almost everywhere convergence of $\nabla v_j(x) \to \nabla v(x)$ (recall Proposition 2.1), (1.5) and Lebesgue's dominated convergence theorem we infer that

$$\int_{B_R^+}\left(\frac{|\nabla v_j|^2}{2} + \mathcal{B}_{\varepsilon_j}(v)\right) \to \int_{B_R^+}\left(\frac{|\nabla v|^2}{2} + \gamma(x)\right).$$

On the other hand by Fubini's theorem

$$\int_S^R \frac{d\rho}{\rho^n}\int_{\partial B_\rho\cap\{x_n>0\}}\left(\nabla v_j\cdot\nu - \frac{v_j}{\rho}\right)^2 = \int_{(B_R\setminus B_S)\cap\{x_n>0\}}\left(\nabla v_j(x)\cdot\frac{x}{|x|} - \frac{v_j(x)}{|x|}\right)^2\frac{dx}{|x|^n}$$

and again using almost everywhere convergence of $\nabla v_j(x) \to \nabla v(x)$, (1.5) and Lebesgue's dominated convergence theorem we infer that

$$\begin{aligned}\int_{(B_R\setminus B_S)\cap\{x_n>0\}}&\left(\nabla v_j(x)\cdot\frac{x}{|x|} - \frac{v_j(x)}{|x|}\right)^2\frac{dx}{|x|^n} \longrightarrow\\ \longrightarrow \int_{(B_R\setminus B_S)\cap\{x_n>0\}}&\left(\nabla v(x)\cdot\frac{x}{|x|} - \frac{v(x)}{|x|}\right)^2\frac{dx}{|x|^n} =\\ &= \int_S^R\frac{d\rho}{\rho^n}\int_{\partial B_\rho\cap\{x_n>0\}}\left(\nabla v\cdot\nu - \frac{v}{\rho}\right)^2.\end{aligned}$$

Consequently, we have $W_\varepsilon \to W$ and in view of (2.5) W is nondecreasing function of r. □

Remark 2.7. *Actually one can say more about $\gamma(x)$ in Lemma 2.6. Namely, there is a function $M(x)$ such that*

$$\gamma(x) = M\chi_{\{v>0\}} + M(x)\llcorner\partial\{v>0\}, \quad 0 \le M(x) \le M \tag{2.5}$$

see [3] (3.7) page 726. Clearly we can take

$$\gamma(x) = M\chi_{\{v>0\}} \quad \text{in } W \qquad \text{if } \mathcal{L}^n(\partial\{v>0\}) = 0, \tag{2.6}$$

where $\mathcal{L}^n$ is the $n-$dimensional Lebesgue measure. This is certainly true for linear function $v = Cx_n$, $C > 0$.

We close this section by proving a simple convergence result for the function $\gamma(x)$ defined in (2.5).

Lemma 2.8. *Let v be a limit of singular perturbation problem. Consider the scaled functions $v_j(x) = \frac{v(r_j x)}{r_j}$, $r_j \to 0$ and v_0 be a blow-up limit corresponding to $\{r_j\}$. Then the functions*

$$\gamma_j(x) = M\chi_{\{v_j>0\}} + M(r_j x)\llcorner\partial\{v_j>0\} \in L^\infty(\mathbb{R}^n_+)$$

weak-star converge to γ_0 in $L^1(B_R^+)$ for any fixed $R > 0$, where

$$\gamma_0(x) = M\chi_{\{v_0>0\}} + M_0(x)\llcorner\partial\{v_0>0\}, \quad 0 \le M_0(x) \le M. \tag{2.7}$$

Proof. Let $x \in \mathbb{R}^n_+$ such that $v_0(x) > 0$ then $v_j(y) \ge \frac{v_0(x)}{2}$ for all $y \in B_\delta(x)$ provided that δ is small. Therefore $\gamma_j = M$ in $B_\delta(x)$ for j large. Analogously, if $v_0(x) < 0$ then $\gamma_j = 0$ in a neighbourhood of x if j is large enough. These imply that

$$\gamma_j(x) \to M\chi_{\{v_0>0\}} \quad \text{weak-star in} \quad L^\infty(B_R^+ \cap \{v_0 \neq 0\}).$$

The fact that $0 \le \gamma_j \le M$ (see (2.5) and the discussion preceding it) implies that there exists $0 \le \gamma_0 \le M$ such that

$$\gamma_j(x) \to \gamma_0(x) \quad \text{weak-star in} \quad L^\infty(B_R^+).$$

Then necessarily $\gamma_0(x) = M\chi_{\{v_0>0\}}$ in $B_R^+ \cap \{v_0 \neq 0\}$, and consequently there is $0 \le M_0(x) \le M$ such that (2.7) hods. $\square$

Remark 2.9. *The conclusion of Lemma* (2.8) *remains true if we consider the blow down limit instead, i.e. if we let $r_j \to \infty$.*

3. Proof of Theorem 1

Proof. It suffices to show that for any $\delta > 0$ there is $r_0, \varepsilon_0 > 0$ depending on f, n, M and L such that

$$\partial\{u^\varepsilon > 0\} \cap B_r^+ \subset \{x_n < \delta|x|\} \cap B_r^+, \quad \forall r < r_0 \text{ and } \quad \varepsilon < \varepsilon_0. \tag{3.1}$$

Suppose, towards a contradiction, that (3.1) fails, then for some fixed δ_0 there is $\{r_j\}_{j=1}^\infty, \{\varepsilon_j\}_{j=1}^\infty$, and $x_j \in \partial\{u^{\varepsilon_j} > 0\} \cap \partial B_{r_j}^+$ such that

$$x_j \in \{x_n > \delta_0|x|\} \cap \partial B_{r_j}^+ \quad \text{and } u^{\varepsilon_j}(x_j) = 0 \tag{3.2}$$

introduce the scaled function

$$v_j(x) = \frac{u^{\varepsilon_j}(r_j x)}{r_j}, \quad x \in B_{\frac{1}{r_j}}^+ \tag{3.3}$$

then we have from (1.5) that

$$|\nabla v_j(x)| \le L, \quad x \in B_{\frac{1}{r_j}}^+ \tag{3.4}$$

and

$$\Delta v_j(x) = \frac{r_j}{\varepsilon_j}\beta\left(\frac{v_j(x)}{\varepsilon_j/r_j}\right) \quad x \in B_{\frac{1}{r_j}}^+. \tag{3.5}$$

Furthermore, in view of (1.2) it follows that the corresponding scaled boundary data is

$$f_j(x) := \frac{f(r_j x)}{r_j} = o(r_j) \quad \text{and} \quad \|f_j(x)\|_{C^{1,\alpha}(\overline{B_M^+})} \to 0 \tag{3.6}$$

for any fixed $M > 0$. Observe that (3.2) translates to the limit configuration such that

$$\exists\, x_0 \in \{x_n > \delta_0|x|\} \cap \partial B_1^+ \quad \text{and } v(x_0) = 0. \tag{3.7}$$

There are three possible scenarios.

Case 1: There is a subsequence, still denoted j, such that $\frac{r_j}{\varepsilon_j} \to 0$. It follows from (3.5) that $0 \le \Delta v_j \le \sup \beta \frac{r_j}{\varepsilon_j} \to 0$. Thus owing to (3.4), (3.6) and Proposition 2.1 it follows that $v_j \to v$ such that v solves the following problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^n_+, \\ |\nabla v(x)| \le L & x \in \mathbb{R}^n_+, \\ v = 0 & \text{on } \{x_n = 0\}. \end{cases}$$

From Liouville's theorem it follows that v is linear, which contradicts (3.7).

Case 2: There is a subsequence, still denoted j, such that $\frac{r_j}{\varepsilon_j} \to a > 0$. Without loss of generality we assume that $a = 1$. From (3.5) and Proposition 2.1 it follows that we can extract a subsequence $j(k)$ such that $v_{j(k)} \to v$ and $\Delta v = \beta(v)$. Thus applying Lemma 2.3 we see that $v > 0$ which is again in contradiction with (3.7).

Case 3: There is a subsequence, still denoted j, such that $\frac{r_j}{\varepsilon_j} \to \infty$. Introduce $\varepsilon'_j = \frac{\varepsilon_j}{r_j} \to 0$ then from (3.5) we get

$$\Delta v_j(x) = \beta_{\varepsilon'_j}(v_j(x)) \quad x \in B^+_{\frac{1}{r_j}}. \tag{3.8}$$

Observe that the boundary data for v_j is determined by $g_{\varepsilon'_j}(x) = f_j(x)$ and

$$\|g_{\varepsilon'_j}\|_{C^{1,\alpha}(B_{r_0}\cap\{x_n=0\})} \to 0$$

thanks to (3.6). Therefore by Lemma 2.6 we have for the limit function $v = \lim_{j\to\infty} v_j$

$$W(r) = \frac{1}{r^n}\int_{B_r^+}\left(\frac{|\nabla v|^2}{2} + \gamma(x)\right) - \frac{1}{2r^{n+1}}\int_{\partial B_r^+\cap\{x_n>0\}} v^2 \tag{3.9}$$

is a monotone function of r, see (2.4). Let us now denote by W^r the functional in (3.9), where γ has been replaced by $\gamma(rx)$, i.e.

$$\begin{aligned} W(s\rho) &= \frac{1}{(s\rho)^n}\int\limits_{B_{s\rho}^+}\left(\frac{|\nabla v|^2}{2} + \gamma(x)\right) - \frac{1}{2(s\rho)^{n+1}}\int\limits_{\partial B_{s\rho}^+\cap\{x_n>0\}} v^2 \\ &= \frac{1}{s^n}\int\limits_{B_s^+}\left(\frac{|\nabla v_\rho|^2}{2} + \gamma(\rho x)\right) - \frac{1}{2s^{n+1}}\int\limits_{\partial B_s^+\cap\{x_n>0\}} v_\rho^2 \\ &=: W^\rho(s,0,v_\rho) \end{aligned} \tag{3.10}$$

and W^0 the corresponding functional with γ_0 in Lemma 2.8. Observe that by (3.10) one has $W^\rho(s,0,v_\rho) \geq W^\rho(\tau,0,v_\rho)$ for $s \geq \tau$.

Then scaling the functional, and using monotonicity, we have

$$\begin{aligned} W(0^+,0,v) &= \lim_{r\to 0} W^r(rs,0,v) = \lim_{r\to 0} W^r(s,0,v_r) = W^0(s,0,v_0) = \\ &= \frac{1}{s^n}\int\limits_{B_s^+}\left(\frac{|\nabla v_0|^2}{2} + \gamma_0(x)\right) - \frac{1}{2s^{n+1}}\int\limits_{\partial B_s^+\cap\{x_n>0\}} v_0^2 \end{aligned}$$

which implies that the blow-up is homogeneous of degree one. In a similar way we conclude that the blow-down v_∞ is homogeneous of degree one, since

$$\begin{aligned}
0 = W(0^+,0,v) - W(0^+,0,v) \quad &\longleftarrow \quad W(rs,0,v) - W(r\tau,0,v) = \\
&= \quad \int_{r\tau}^{rs} \frac{d\rho}{\rho^n} \int_{\partial B_\rho \cap \{x_n>0\}} \left(\nabla v \cdot \nu - \frac{v}{\rho}\right)^2 \\
&= \quad \int_{\tau}^{s} \frac{d\rho}{\rho^n} \int_{\partial B_\rho \cap \{x_n>0\}} \left(\nabla v_r \cdot \nu - \frac{v_r}{\rho}\right)^2 \\
&\longrightarrow \quad \int_{\tau}^{s} \frac{d\rho}{\rho^n} \int_{\partial B_\rho \cap \{x_n>0\}} \left(\nabla v_0 \cdot \nu - \frac{v_0}{\rho}\right)^2,
\end{aligned}$$

for $s \geq \tau$.

Applying Lemma 2.2 to the limit and its even reflection across $x_n = 0$ we see that $v \geq 0$ and hence v_0 (the blow-up) and v_∞ (the blow down) must be linear functions. Thus

$$v_0(x) = C_0 x_n \text{ and } v_\infty(x) = C_\infty x_n. \tag{3.11}$$

Now it remains to check that $C_0 = C_\infty$. It suffices to show that v is homogeneous function of degree one. From upper-semi continuity (see Claim 4.2 in Appendix) it follows that the energy at the origin is larger than or equal to the energy at all other free boundary points in a small neighborhood. In particular if we take a sequence of regular free boundary points in $\mathbb{R}^n_+$, where $|\nabla u(z)| = \sqrt{2M}$, the energy is a fixed constant A, i.e.

$$\lim_{z\to 0} W(0^+, z, v) =: A,$$

at any regular free boundary point z (with $z_1 > 0$) where one necessarily has $|\nabla u(z)| = \sqrt{2M}$, which is simply the free boundary condition satisfied at regular points in classical sense.

Thus we obtain for $z \in \partial\{v > 0\}$ with $z_n > 0$

$$
\begin{aligned}
A &= W(0^+, z, v) = \lim_{r_j \downarrow 0} W(r_j, z, v) \\
&= \lim_{r_j \downarrow 0} \frac{1}{r_j^n} \int_{B_{r_j}(z) \cap \{x_n > 0\}} \left(\frac{|\nabla v|^2}{2} + \gamma(x) \right) - \frac{1}{2 r_j^{n+1}} \int_{\partial B^+_{r_j}(z) \cap \{x_n > 0\}} v^2 \\
&= \lim_{j \to \infty} \int_{B_1} \left(\frac{|\nabla v_j(y)|^2}{2} + \gamma(z + r_j y) \right) - \frac{1}{2} \int_{\partial B_1} v_j^2
\end{aligned}
$$

where we set $v_j(y) = \frac{v(z+r_j y)}{r_j}$, and in the second equality we have used the fact that in $B_{r_1}(z)$, for some small r_1, there is no fixed boundary presented, and hence the situation is like an interior case for the monotonicity function. Setting $\gamma_j(y) = \gamma(z + r_j y)$ and recalling Lemma 2.8 we see

$$
\begin{aligned}
A &= \int_{B_1} \left(\frac{|\nabla v_0|^2}{2} + \gamma_0 \right) - \frac{1}{2} \int_{\partial B_1} v_0^2 \\
&= \frac{1}{2} \int_{\partial B_1} v_0 \nabla v_0 \cdot \nu + \int_{B_1} \gamma_0 - \frac{1}{2} \int_{\partial B_1} v_0^2
\end{aligned}
\tag{3.12}
$$

where to get the last line we used the divergence theorem, i.e. $\mathrm{div}(v_0 \nabla v_0) = |\nabla v_0|^2 + v_0 \Delta v_0 = |\nabla v_0|^2$ because $v_0 = c_0 (x \cdot e)^+$ for some fixed unit direction e (recall that v_0 is a blow-up of v at a regular free boundary point z). Consequently, v_0 is homogeneous function of degree one i.e. $v_0(x) = \nabla v_0(x) \cdot x$ which gives that $\frac{1}{2} \int_{\partial B_1} v_0 \nabla v_0 \cdot \nu - \frac{1}{2} \int_{\partial B_1} v_0^2 = 0$. Returning to (3.12) we infer from (2.7) with $v_0 = c_0 (x \cdot e)^+$ and (2.6)

$$
\begin{aligned}
A &= \int_{B_1} \gamma_0 \\
&= M \chi_{\{v_0 > 0\}}.
\end{aligned}
\tag{3.13}
$$

From upper semicontinuity (Claim 4.2 in Appendix) we see that

$$
W(0^+, 0, v) \geq \lim_{z \to 0} W(0^+, z, v) = A.
$$

This in conjunction with the monotonicity of W (2.4) and scaling property of W (3.10) yield

$$\begin{aligned} A &\le W(0^+,0,v)\le \\ &\le W(\infty,0,v)=\lim_{j\to\infty}W(r_j,0,v)=\lim_{j\to\infty}W^{r_j}(1,0,v_j)= \\ &= \lim_{j\to\infty}\left[\int\limits_{B_1^+}\left(\frac{|\nabla v_j|^2}{2}+\gamma(r_jx)\right)-\int\limits_{\partial B_1^+\cap\{x_n>0\}}v_j^2\right] \\ &= \int_{B_1^+}\left(\frac{|\nabla v_\infty|^2}{2}+\gamma_\infty\right)-\frac{1}{2}\int_{\partial B_1^+}v_\infty^2 \end{aligned}$$

where $v_j(x)=\frac{v(r_jx)}{r_j}$ and γ_∞ is the limit of the functions $\gamma_j(r_jx)$ as $r_j\to\infty$.

By (3.11) $v_\infty=C_\infty x_n, x\in\mathbb{R}^n_+$. In view of (2.6) it follows that $\gamma_\infty = M\chi\{u_\infty>0\}$. Thus using the divergence theorem we conclude

$$\begin{aligned} A &\le \frac{1}{2}\int_{\partial B_1^+}v_\infty\nabla v_\infty\cdot\nu-\frac{1}{2}\int_{\partial B_1^+}v_\infty^2+\int_{B_1^+}M\chi\{u_\infty>0\} \\ &= \int_{B_1^+}M\chi\{u_\infty>0\}=A. \end{aligned} \tag{3.14}$$

Since by monotonicity

$$A\le W(0^+,0,v)\le W(r,0,v)\le W(\infty,0,v)\le A$$

we see that, in view of (2.4), v is homogeneous. Applying Lemma 2.2 with $h_1=v$ and h_2 being the even reflection of v across $x_n=0$, we see that v must be linear. But being linear it has to be cx_n and hence this is in contradiction with (3.7). □

4. Appendix

We define the subset of free boundary points T, which are touching the hyperplane $\{x_n=0\}$ as follows:

$$T=\{z:\ \exists\, z^j\in\partial\{v>0\},\ z^j_n>0,\ z^j\to z\},\tag{4.1}$$

where $\partial\{v>0\}$ denotes free boundary points.

In what follows we set

$$W(\rho,z,v)=\frac{1}{\rho^n}\int_{B_\rho(z)\cap\{x_n>0\}}\frac{|\nabla v|^2}{2}+\gamma(x)-\frac{1}{2\rho^{n+1}}\int_{\partial B_\rho(z)\cap\{x_n>0\}}v^2\tag{4.2}$$

where $\gamma(x)$ is defined by (2.5) and $z \in \overline{\mathbb{R}^n_+} \cap \partial\{v > 0\}$.

Claim 4.1. *For any given $0 < \tau < 10^{-5}$, $k > 10^5$ there exists $R_0 > 0$ such that if for some free boundary point z with $z_1 > 0$ and $dist(z, T) := r_0 < R_0$ (see (4.1) for the definition of the class of touching free boundary points) then*

$$W(r_0, z, v) \leq (1 + \tau)W(kr_0, z, v).$$

Proof. If the claim fails then there are $k > 10^5, 0 < \tau < 10^{-5}$ and a sequence $r_j \to 0$, and $x^j \to x_0 \in T$, with $dist(x^j, T) = r_j$ and

$$W(r_j, x^j, v) \geq (1 + \tau)W(kr_j, x^j, v). \tag{4.3}$$

Without loss of generality we take $x_0 = 0$. Scale the solution and the monotonicity function with r_j, that is

$$u_j(x) := \frac{v(x_j + r_j x)}{r_j},$$

so (4.3) translates to

$$W(1, 0, v_j) \geq (1 + \tau)W(k, 0, v_j).$$

Letting $r_j \to 0$ and nothing that

$$\frac{x_n^j}{r_j} \to 0 \quad \text{(because of tangential touch)}$$

we have that $w = \lim v_j$ satisfies

$$W(1, 0, w) \geq (1 + \tau)W(k, 0, w)$$

where now w is a global solution over $\mathbb{R}^n_+$ because $x_n^j/r_j \to 0$. Since also $w(x', 0) = 0$, we arrive at a contradiction to the monotonicity formula in Lemma 2.6. This proves the claim. $\square$

Claim 4.2. *The function $\omega(z) := W(0^+, z, v)$ is upper-semi continuous.*

Proof. Choose z on the free boundary, with $z_n > 0$, then $\omega(z) \le W(r_0, z, v)$ for $r_0 \le dist(z, T)$, because there is no touching point in this radius, and the problem is like an interior problem.

Next by Claim 4.1, we have

$$\begin{aligned} \omega(z) &\le W(r_0,z,v) \le (1+\tau)W(kr_0,z,v) \le \\ &\le (1+\tau)\left[\left(\frac{k+1}{k}\right)^n W((k+1)r_0,0,v) + I_k(z)\right], \end{aligned} \tag{4.4}$$

where

$$I_k(z) = \frac{1}{2k^n(k+1)r_0^{n+1}}\int_{\partial B^+_{(k+1)r_0}(0)\cap\{x_n>0\}} v^2 - \frac{1}{2(kr_0)^{n+1}}\int_{\partial B^+_{kr_0}(z)\cap\{x_n>0\}} v^2.$$

We wish to estimate I_k as follows

$$\begin{aligned} 2(r_0k)^{n+1}I_k &= \frac{k}{k+1}\int_{\partial B^+_{(k+1)r_0}(0)\cap\{x_n>0\}} v^2 - \int_{\partial B^+_{kr_0}(z)\cap\{x_n>0\}} v^2 \\ &= \frac{k}{k+1}\left(\frac{k+1}{k}\right)^{n-1}\int_{\partial B^+_{kr_0}(0)\cap\{x_n>0\}} v^2((1+1/k)y) - \\ &\quad - \int_{\partial B^+_{kr_0}(z)\cap\{x_n>0\}} v^2 \\ &= \left(\frac{k+1}{k}\right)^{n-2}\int_{\partial B^+_{kr_0}(0)\cap\{x_n>0\}} v^2((1+1/k)y) - \\ &\quad - \int_{\partial B^+_{kr_0}(z)\cap\{x_n>0\}} v^2 \\ &= \left(\frac{k+1}{k}\right)^{n-2}\int_{\partial B^+_{kr_0}(0)\cap\{x_n>0\}} v^2((1+1/k)y) - \\ &\quad - \int_{\partial B^+_{kr_0}(z)\cap\{0<x_n<z_n\}} v^2 - \int_{\partial B^+_{kr_0}(z)\cap\{x_n>z_n\}} v^2 \\ &= \left(\frac{k+1}{k}\right)^{n-2}\int_{\partial B^+_{kr_0}(0)\cap\{x_n>0\}} v^2((1+1/k)y) \\ &\quad - \int_{\partial B^+_{kr_0}(z)\cap\{x_n>z_n\}} v^2 - O((kr_0)^{n-2}(z_n)^3) \end{aligned}$$

$$= \left(1+\frac{1}{k}\right)^{n-2}\int_{\partial B^+_{kr_0}(0)\cap\{x_n>0\}} v^2((1+1/k)y) - \tag{4.5}$$

$$- \int_{\partial B^+_{kr_0}(z)\cap\{y_n>0\}} v^2(z+y) - O((kr_0)^{n-2}(z_n)^3).$$

Notice that by tangential touch

$$O((kr_0)^{n-2}(z_n)^3) = (kr_0)^{n-2} o(r_0^3). \tag{4.6}$$

Furthermore

$$\begin{aligned}
|v^2((1+1/k)y) - v^2(z+y)| &= |v((1+1/k)y) - v(z+y)||v((1+1/k)y) + v(z+y)| \\
&\leq L^2 \left[\left(\frac{|y|}{k} + |z| \right) \left((2+\frac{1}{k})|y| + |z| \right) \right] \\
&\leq L^2 \left[(r_0 + |z|) \left((2+\frac{1}{k}) kr_0 + r_0 \right) \right] \\
&\leq 2(2k+2) L^2 r_0^2 \\
&= 4(k+1) L^2 r_0^2
\end{aligned}$$

where L is the Lipschitz constant defined in (1.5).

Combining this with (4.5) and (4.6) we obtain

$$|I_k| \leq \frac{O(1)}{k} + \frac{2(k+1)L^2}{k^2} + \frac{o(r_0^3)}{(kr_0)^3}.$$

Returning to (4.4) and letting $z \to T$, i.e. $r_0 \to 0$ we arrive at

$$\lim_{z\to 0} \omega(z) \leq (1+\tau) \left(\frac{k+1}{k} \right)^n \omega(0) + \frac{C}{k}.$$

Since by construction ε is any number in the interval $(0, 10^{-5})$ and k is any number such that $k > 10^5$ (see the statement of Claim 4.1) we first send $k \to \infty$ and then $\tau \to 0$ to conclude the desired result. □

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